ACTEX Academic Series

Solutions to Theory of Interest and Life Contingencies with Pension Applications: A Problem-Solving Approach

Fourth Edition

4th

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CHAPTER ONE

1. **a.** $14{,}000\,[1+6(.03)] = 16520.00$

b. $14{,}000(1.03)^6 = 16{,}716.73$

c. $14{,}000\left[1+\frac{66}{365}(.03)\right] = 14{,}075.95$

d. $14{,}000\left[\frac{318}{365}(1.03)^2+\frac{47}{365}(1.03)^3\right] = 14909.98$

2. **a.** $14{,}000(1.038)^5 = 16869.99$

b. $14{,}000\left[1+5\frac{96}{365}(.038)\right] = 16799.92$

c. $14{,}000\left[\frac{269}{365}(1.038)^5+\frac{96}{365}(1.038)^6\right] = 17038.60$

3. $a(t)=t^2+t+1$, and $i_n=\frac{2n}{n^2-n+1}$ from part (e) of Example 1.1.

Then $i_{n+1}=\frac{2(n+1)}{(n+1)^2-(n+1)+1}=\frac{2n+2}{n^2+n+1}$. Assume $i_{n+1}\geq i_n$. Then $\frac{2n+2}{n^2+n+1}\geq\frac{2n}{n^2-n+1}$, or $(2n)(n^2-n+1)\geq 2n(n^2+n+1)$, or $(2n^3+2)\geq(2n^3+2n^2+2n)$, which is clearly false for all $n\geq 1$.

4. **a.** $a(0)=\sqrt{1+(i^2+2i)0}=1$; $a(1)=\sqrt{1+i^2+2i}=1+i$.

b. $\frac{d}{dt}a(t)=\frac{1}{2}\left[1+(i^2+2i)t^2\right]^{-1/2}\cdot 2t=\frac{t}{[1+(i^2+2i)t^2]^{1/2}}>0$ for $t>0$, so $a(t)$ is increasing. Furthermore, root functions are continuous in their domain.

c. $a(t)=\sqrt{1+(i^2+2i)t^2}\gtreqless 1+it$ according as $1+(i^2+2i)t^2\gtreqless 1+2it+i^2t^2$, or according as $2it^2\gtreqless 2it$, or according as $t\gtreqless 1$. Then $a(t)>1+it$ if $t>1$, but $a(t)<1+it$ if $t<1$.

d. Consider

$$\Delta(t)=(1+i)^t-a(t)=(1+i)^t-\left[1+(i^2+2i)t^2\right]^{1/2}.$$

We find

$\frac{d}{dt}\Delta(t)=(1+i)^t\cdot\ln(1+i)-\frac{1}{2}\left[1+(i^2+2i)t^2\right]^{-1/2}\cdot 2(i^2+2i)t$

$=(1+i)^t\cdot\ln(1+i)-\frac{(i^2+2i)}{[1/t^2+(i^2+2i)]^{1/2}}$.

Now as t gets larger, the negative term approaches the constant $(i^2+2i)^{1/2}$. Since $(1+i)^t$ increases, then eventually $(1+i)^t\cdot\ln(1+i)$ exceeds the negative term, so $\frac{d}{dt}\Delta(t)$ is eventually positive, so $\Delta(t)$ is eventually increasing, so $(1+i)^t$ eventually exceeds $a(t)$.

5. **a.** If $1+i_n = \frac{a(n)}{a(n-1)}$ is constant, then $\ln(1+i_n) = \ln[a(n)] - \ln[a(n-1)] = \Delta \ln[a(n-1)]$ is constant as well. By the properties of finite differences, if $\Delta f(n)$ is constant, then $f(n)$ itself is a linear function. Thus we find $\ln[a(n-1)]$, and hence $\ln[a(n)]$ to be a linear function, so $a(n) = e^{\ln[a(n)]}$ is an exponential function, which can be put in the form $(1+i)^t$.

b. No. The argument in (a) assumed integral values of t to establish the properties of the Δ operator.

6. **a.** Since $A(n-1)$ is the value of an investment at time $n-1$, and $A(n)$ is its value at time n, the difference, I_n, is the growth due to interest in the n^{th} time period.

b. $A(n) - A(0) = [A(n) - A(n-1)] + [A(n-1) - A(n-2)] + \cdots + [A(1) - A(0)] = I_n + I_{n-1} + \cdots + I_1$

c. The result is intuitive. Clearly $A(n) - A(0)$ is the total interest earned over n periods, which is the sum of the interest earned in each period.

d. $i_1 + i_2 + \cdots + i_n = \frac{a(1)-a(0)}{a(0)} + \frac{a(2)-a(1)}{a(1)} + \cdots + \frac{a(n)-a(n-1)}{a(n-1)} \neq a(n) - a(0)$,

so no. The i_r are rates, not amounts, so they are not additive to any meaningful concept.

7. **a.** $1000[1+.04t] = 1400$, so $t = \frac{\frac{14}{10}-1}{.04} = 10$.

b. $1000(1+12i) = 1500$, so $i = \frac{\frac{15}{10}-1}{12} = 4\frac{1}{6}\%$.

c. $1000(1.04)^t = 1400$, so $t = \frac{\log 1.40}{\log 1.04} = 8.57894$.

$1000(1+i)^{12} = 1500$, so $i = (1.5)^{1/12} - 1 = 3.4366\%$.

8. $1000(1+it) = 1060$ implies $it = .06$. Then

$500\left[1 + (\tfrac{2}{3}i)(2t)\right] = 500\left[1 + \tfrac{4}{3}it\right] = 500\left[1 + (\tfrac{4}{3})(.06)\right] = 540$.

9. $6000(1.03)^4(1.042)^6 = 8643.83$

10. $(1.043)(1.037)(1.05) = (1+i)^3$, so

$i = [(1.043)(1.037)(1.05)]^{1/3} - 1 = 4.332\%$.

11. We are given $(1+i)^x = 2$, $2(1+i)^y = 3$, and $(1+i)^z = 5$.

In $z - x - y$ years, 6 will grow to

$6(1+i)^{z-x-y} = \frac{6(1+i)^z}{(1+i)^x(1+i)^y} = \frac{(6)(5)}{(2)(1.5)} = 10$.

12. We wish to prove that if $(1+i)^x = (1+2i)^y$, then $x < 2y$ for all i, $i > 0$. Assume $x \geq 2y$. Then $(1+i)^x \geq (1+i)^{2y}$. But $(1+i)^x = (1+2i)^y$, so our implication is $(1+2i)^y \geq (1+i)^{2y}$, or

$1 + 2i(y) + \frac{y(y-1)}{2}4i^2 + \frac{y(y-1)(y-2)}{6}i^3 + \cdots$

$\geq 1 + 2yi + \frac{2y(2y-1)}{2}i^2 + \frac{2y(2y-1)(2y-2)}{6}i^3 + \cdots$

or $\frac{y}{2}i^2(4y-4) + \frac{y}{6}i^3(8y^2 - 24y + 16) + \cdots \geq \frac{y}{2}i^2(4y-2)$

$+\frac{y}{6}i^3(8y^2 - 12y + 4) + \cdots$

But $(4y-4)$ is less than $(4y-2)$, so we reach a contradiction, implying $x < 2y$. Therefore statement (a) is true.

13. **a.** $PV = 1000(1-d)^3 = 884.74$

b. $(1+i)^{-1} = 1 - d = .96$, so $i = (.96)^{-1} - 1 = 4.1667\%$

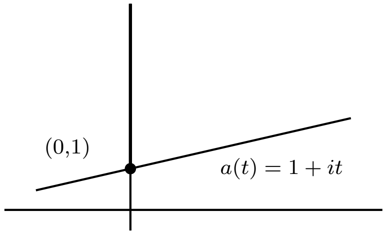
c. $PV = 1000(1+i)^{-3} = 884.74$

14. **a.** $PV = 14{,}000(1.04)^{-6} = 11064.40$

b. $PV = 14{,}000(.96)^6 = 10958.61$

c. A rate of discount equivalent to a rate of interest is numerically smaller than that rate of interest. If the rates are numerically the same, then the discount is greater than the equivalent interest, so it produces a smaller discounted value.

15. **a.**



b. The present value t years in the past is $(1+it)^{-1}$; $1 - it$ will turn negative in $\frac{1}{i}$ years.

16. $d_n = \frac{a(n)-a(n-1)}{a(n)} = \frac{(1+i)^n-(1+i)^{n-1}}{(1+i)^n} = 1 - \frac{1}{1+i}$, which is constant.

17. **a.** Starting with (1.16), $d = \frac{i}{1+i} = i(\frac{1}{1+i}) = iv$. Verbally, the discount on one unit of money is the present value of the interest on that unit.

b. Since $v = \frac{1}{1+i}$, then $i = \frac{1}{v} - 1$, so $d = iv = (\frac{1}{v} - 1)v = 1 - v$. Verbally, the discount on one unit is equal to the unit minus its discounted value.

c. $i - d = i - iv = i(1 - v) = id$. Verbally, the difference between the interest and the discount on a unit of money is the discount on the interest itself.

d. Since $d = \frac{i}{1+i}$, then $\frac{1}{d} = \frac{1+i}{i}$, and $\frac{1}{d} - \frac{1}{i} = \frac{i}{i} = 1$.

e. $d\left(1 + \frac{i}{2}\right) = d + \frac{id}{2} = d + \frac{i-d}{2} = \frac{i}{2} + \frac{d}{2}$; similarly,
$i\left(1 - \frac{d}{2}\right) = i - \frac{id}{2} = i - \frac{i-d}{2} = \frac{i}{2} + \frac{d}{2}$.

f. $i(1-d)^{1/2} = \frac{d}{1-d}(1-d)^{1/2} = d(1-d)^{-1/2} = d(1+i)^{1/2}$

18. **a.** $\frac{d^3}{(1-d)^2} = \frac{d^3}{v^2} = \frac{d^3}{(\frac{d}{i})^2}$ (since $d = iv$) $= i^2 d$

b. $\frac{(i-d)^2}{1-v} = \frac{(id)^2}{d} = i^2 d$

c. $(i-d)d = (id)d = id^2$, which is the exception

d. $i^3 - i^3 d = i^3(1-d) = i^3 v = i^3(\frac{d}{i}) = i^2 d$

e. Already is $i^2 d$

19. $d = \frac{i}{1+i}$, and we are given $i = \frac{210}{L}$ and $dL = 200$.

Substituting, we have $\frac{210/L}{1+210/L} \cdot L = 200, or\, \frac{210 \cdot L}{L+210} = 200$,

or $210L = 200L + (200)(210)$. Then $L = \frac{(200)(210)}{10} = 4200$.

20. We will calculate equivalent effective annual rates:

A: $i = (1.01025)^4 - 1 = .041635$

B: $i = (1 + \frac{.04096}{5})^5 - 1 = .041637$

C: $i = (1 - \frac{.04064}{10})^{-10} - 1 = .041563$

Thus B is the most advantageous to the investor (largest effective annual rate), whereas C is most advantageous to Acme Trust (smallest effective annual rate).

21. $AV = 3000(1.009)^{11} = 3310.73$

22. **a.** $(1+i) = (1.024)^2$, so $i = (1.024)^2 - 1 = 4.8576\%$

b. $\left[1 - \frac{d^{(4)}}{4}\right]^{-4} = (1.024)^2$, so $d^{(4)} = 4\left[1 - (1.024)^{-1/2}\right] = 4.7153\%$

c. $\left[1 + \frac{i^{(12)}}{12}\right]^{12} = (1.024)^2$, so $i^{(12)} = 12\left[(1.024)^{1/6} - 1\right] = 4.7527\%$

d. $(1+j)^6 = 1.024$, so $j = (1.024)^{1/6} - 1 = .396\%$

23. **a.** We are given $100(1+j) = 102.50$, so $j = .025$ effective per half-year.

b. $i^{(2)} = 2j = 5\%$

c. $i = (1.025)^2 - 1 = 5.0625\%$

d. $\left[1 - \frac{d^{(3)}}{3}\right]^{-3} = (1.025)^2$, so $d^{(3)} = 3\left[1 - (1.025)^{-2/3}\right] = 4.8981\%$

24. $150 = 20\left(1+\dfrac{d}{4}\right)^{4\cdot 15}\left(1+\dfrac{.04}{2}\right)^{2\cdot 25} + 30\left(1+\dfrac{.04}{2}\right)^{2\cdot 20}$

$$= 20\left(1+\frac{d}{4}\right)^{60}(1.02)^{50} + 30(1.02)^{40}$$

$$= 53.8318\left(1+\frac{d}{4}\right)^{60} + 66.2412$$

$\left(1+\frac{d}{4}\right)^{60} = \frac{150-53.8318}{66.2412} = 1.45179$

$1 + \frac{d}{4} = 1.0062326$

$\therefore d = .02493$

25. Dara: $500\left(1+\frac{i}{4}\right)^{4\cdot 11}\left(\frac{i}{4}\right)$

Pramila: $1000\left(\frac{i}{4}\right)$

$500\left(1+\frac{i}{4}\right)^{4\cdot 11}\left(\frac{i}{4}\right) = 1000\left(\frac{i}{4}\right)$ (assume $i > 0$)

$\left(1+\frac{i}{4}\right)^{44} = 2 \Rightarrow i = 4\left(2^{\frac{1}{44}} - 1\right) = 4(.015878) = 0.06351$

26. $\left[1 + \frac{i^{(n)}}{n}\right] = (1+i)^{1/n} = \frac{(1+i)^{1/6}}{(1+i)^{1/8}} = (1+i)^{1/24}$. Therefore $\frac{1}{n} = \frac{1}{24}$, so $n = 24$.

27. $\left[1 - \frac{d^{(7)}}{7}\right]^{-7} = \left[1 + \frac{i^{(5)}}{5}\right]^{5}$, so $d^{(7)} = 7\left[1 - \left(1 + \frac{i^{(5)}}{5}\right)^{-5/7}\right]$

28. $v\left(1 + \frac{i^{(3)}}{3}\right) = (1-d)(1+i)^{1/3}$

$= (1-d)^{1/2}(1+i)^{-1/2}(1+i)^{1/3}$

$= (1+i)^{-1/6}(1-d)^{1/2}$

similarly,

$\left(1 + \frac{i^{(30)}}{30}\right)\left(1 - \frac{d^{(5)}}{5}\right)(1-d)^{1/2} = (1+i)^{1/30}(1+i)^{-1/5}(1-d)^{1/2}$

$= (1+i)^{-1/6}(1-d)^{1/2}$